

MATRIX FACTORISATIONS ARISING FROM WELL-GENERATED COMPLEX REFLECTION GROUPS

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ABSTRACT. In this note we discuss an interesting duality known to occur for certain complex reflection groups, we prove in particular that this duality has a concrete representation theoretic realisation. As an application, we construct matrix factorisations of the highest degree basic invariant which give free resolutions of the module of Kähler differentials of the coinvariant algebra A associated to such a reflection group. From this one can read off the Hilbert series of $\mathrm{Der}_{\mathbb{C}}(A, A)$. This applies for instance when A is the cohomology of any complete flag manifold, and hence has geometric consequences.

1. INTRODUCTION

To a complex reflection group W acting on a vector space V one may associate the algebra of coinvariants is $A = \mathbb{C}[V]/(\mathbb{C}[V]_+^W)$. By the classical Chevalley-Shephard-Todd theorem A is a zero dimensional complete intersection. As such A enjoys good homological properties. Our aim is to investigate how the context at hand might further influence the homological properties of A . For instance, when W is well-generated we will be able to precisely calculate the dimensions of the graded components of $\mathrm{Der}_{\mathbb{C}}(A, A)$, and indeed of the whole André-Quillen cohomology $H_{AQ}^*(A, A)$.

Let us specialise to the case that W is the Weyl group of a semi-simple algebraic group G . If we choose a Borel subgroup B and a maximal torus T in B , then W acts on $V = \mathrm{Lie}(T)$ as a reflection group. According to the famous “Borel picture” the cohomology of the complete flag manifold $X = G/B$ is canonically isomorphic to the corresponding algebra of coinvariants $A \cong H^*(X; \mathbb{C})$. Because X is a formal space, the graded algebra A completely determines the rational homotopy type of X . For this reason, homological facts about A can be translated into homotopy theoretic information about X .

Specialising further still, one simple question which motivated the constructions below is as follows. The symmetric group $W = \mathfrak{S}_{n+1}$ acts on $S = \mathbb{C}[x_0, \dots, x_n]$ by permuting the variables. The invariant subring S^W is the polynomial algebra $\mathbb{C}[\sigma_1, \dots, \sigma_{n+1}]$ on the elementary symmetric polynomials, so the algebra of coinvariants is $A = S/(\sigma_1, \dots, \sigma_{n+1})$. The module of Kähler differentials $\Omega_{A/\mathbb{C}}^1$ may be presented as the cokernel of the Vandermonde matrix

$$\begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix} : A^{n+1} \rightarrow A^{n+1} \rightarrow \Omega_{A/\mathbb{C}}^1 \rightarrow 0.$$

How can one continue this to a resolution of $\Omega_{A/\mathbb{C}}^1$? Experiments performed by Ragnar-Olaf Buchweitz in Macaulay 2 suggested that the resolution might be 2-periodic. Remarkably, this is always the case: the Vandermonde matrix is part of a matrix factorisation of σ_{n+1} over the algebra $S/(\sigma_1, \dots, \sigma_n)$. This curious

combinatorial fact does not seem to appear in the literature. Knowing this, one can deduce facts about the space $Fl_n(\mathbb{C})$ of complete flags $0 = V^0 \subseteq V^1 \subseteq \dots \subseteq V^n = \mathbb{C}^n$ in \mathbb{C}^n .

At the core of these considerations is a duality discovered by Orlik and Solomon [12] which takes place in the invariant theory of certain complex reflection groups (all Weyl groups included). These are the duality groups, otherwise known as the well-generated reflection groups. We show in section 3 that this duality is realised by a concrete, representation theoretic pairing. In section 5 we use this pairing to construct 2-periodic resolutions of the A modules $\text{Der}_{\mathbb{C}}(A, A)$ of derivations and $\Omega_{A/\mathbb{C}}^1$ of Kähler differentials. Finally, in section 6 we return to our geometric motivation and deduce some facts about flag manifolds. The main arguments will be relatively simple. Almost everything below works for an arbitrary field of characteristic zero, but we will work over \mathbb{C} .

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2. PRELIMINARIES

We begin by reviewing some well-known facts from the invariant theory of complex reflection groups, all of which can be found in [10] or [1]. Let W be a complex reflection group acting irreducibly on a complex vector space V of dimension n . Then W acts on the ring $S = \mathbb{C}[V] = \text{Sym}_{\mathbb{C}}(V^*)$ of polynomials on V . We will grade S by placing V^* in degree 1.¹ If M is a graded vector space $M(n)$ denotes the n -fold shift $M(n)_i = M_{i-n}$.

Let S^W be the algebra of invariants in S . The ideal $I = S \cdot S_+^W$ of S is called the Hilbert ideal, and the quotient $A = S/I = S \otimes_{S^W} \mathbb{C}$ is called the algebra of coinvariants. One special feature of the theory of complex reflection groups is that I has a canonical $\mathbb{C}W$ -module complement in S , defined as follows. The graded dual $S^* \cong \text{Sym}_{\mathbb{C}}(V)$ of S may be thought of as the algebra of differential operators on S , with composition as multiplication. Explicitly, the action $V \otimes S \rightarrow S$ by derivations extends uniquely to a graded left module structure $S^* \otimes S \rightarrow S$. Of course W acts on S^* , and one can define H to be the space of polynomials in S which are killed by every non-constant invariant differential operator in $(S^*)_+^W$. The upshot is that the composition $H \hookrightarrow S \twoheadrightarrow A$ is an isomorphism, see e.g. [10, Corollary 9.37]. Thus H provides canonical representatives for elements of A in S ; it is known as the space of harmonic polynomials².

The ring of invariants is itself a polynomial ring: the canonical map $\text{Hom}_W(V, H) \cong (V^* \otimes H)^W \rightarrow S$ given by multiplying elements of V^* and H induces an isomorphism

$$\text{Sym}_{\mathbb{C}}(\text{Hom}_W(V, H)) \xrightarrow{\sim} S^W.$$

This is one half of the classical Chevalley-Shephard-Todd Theorem³, which says further that complex reflection groups are characterised by this invariant theoretic property. The theorem means we may interpret the quotient $V//W$ as the vector space $\text{Hom}_W(V, H)^*$. An equivalent statement of the theorem is that multiplication $H \otimes S^W \rightarrow S$ is an isomorphism of $\mathbb{C}W$ -modules. It follows that the Hilbert ideal is generated by a regular sequence of length n , and that A is a zero dimensional complete intersection.

¹ This is done to remain consistent with the reflection group literature and its intricate numerology. Motivation from geometry suggests placing V^* in degree 2.

² Everything below would work fine just choosing a fixed W -invariant complement to I in S , we use H only for aesthetic reasons.

³ This is not the usual way of stating the Chevalley-Shephard-Todd Theorem, but it can easily be deduced from it. Note that this way of stating the theorem provides a canonical space of basic invariants.

Note that from the identification $S = H \otimes S^W$ one may deduce that H provides representatives in S for the algebra of functions on any of the fibres of $V \rightarrow V//W$, not just the fibre over 0. In other words $H \hookrightarrow S \rightarrow S \otimes_{S^W} \mathbb{C}_p$ is an isomorphism for any algebra map $p : S^W \rightarrow \mathbb{C}$. In other words H solves the multivariate Lagrange interpolation problem for certain symmetric arrangements of points; the case of a cyclic group acting on \mathbb{C} recovers the discrete Fourier transform.

By a classical theorem of Steinberg [17], the critical locus C of the quotient $V \rightarrow V//W$ is precisely the union of the reflecting hyperplanes. The regular locus is by definition $V^{\text{reg}} = V \setminus C$, elements of V^{reg} are called regular vectors. By Steinberg's theorem there is a natural surjection from the so-called braid group $B(W) = \pi_1(V^{\text{reg}})$ onto $W \cong \pi_1(V^{\text{reg}}//W)$. The image of the critical locus in $V//W$ is called the discriminant locus, and this is cut out by a single reduced polynomial Δ in S^W .

The degrees in which $\text{Hom}_W(V, H)$ is nonzero (counted with multiplicity) are called the degrees of W , denoted by $d_1 \leq \dots \leq d_n$. They are one more than the degrees of the V -isotypic components in H . The codegrees $d_1^* \geq \dots \geq d_n^*$ are by definition the degrees in which $\text{Hom}_W(V^*, H)$ is nonzero, so these are one less than the degrees of the V^* -isotypic components in H . A fixed choice of homogeneous generators of S^W , which necessarily have degrees d_i , is called a set of basic invariants, they are the image of a homogeneous basis of $\text{Hom}_W(V, H)$ under the canonical map to S^W . The highest degree d_n plays an important role, it will be denoted throughout simply by d .

More generally the degrees of a representation M , denoted $d_1^M \leq \dots \leq d_n^M$, are the degrees in which $\text{Hom}_W(M, H)$ is nonzero. The codegrees of M are simply the degrees of M^* .

We say that W is a duality group if it happens that $d_i + d_i^* = d$ for all i . This is a priori a property of A as a graded representation of W alone, but we will see that the commutative algebra of A is heavily influenced by this condition.

We say that W is well-generated if it can be generated by exactly n complex reflections. Classical convex geometry reveals this to be the case for all real reflection groups, that is, those for which the action of W on V is the complexification of an action of W on a real vector space. A real reflection group is precisely a finite Coxeter group equipped with the complexification of its Tits representation.

The following theorem was first observed by Orlik and Solomon in [12] using a case-by-case check of Shephard-Todd classification.

Theorem. *A complex reflection group is a duality group exactly when it is well-generated.*

In [2] Bessis proves that duality groups are well-generated without using the classification. A duality group W possesses an essentially unique highest degree basic invariant f_d . It is not difficult to see (using lemma 1) that in this case a generic line L of direction f_d in $V//W$ will intersect the discriminant hypersurface transversally exactly n times. Bessis shows that after removing these points, the inclusion of the punctured line L' into $V^{\text{reg}}//W$ is surjective on fundamental groups, producing n generators for the braid group, and hence for W . Despite this pleasing argument, there still seems to be no conceptual proof of the converse. The considerations below may shed light on this.

If k is an integer let us denote by I_k^W the ideal in S^W generated by homogeneous polynomials with degree not divisible by k . The corresponding ideal in S is denoted $I_k = S \cdot I_k^W$. In particular $I_0 = I$ is the Hilbert ideal. The notation is consistent in that I_k^W consists exactly of the invariant polynomials in I_k . Note that I_k is generated by a regular sequence, namely, by those basic invariants it contains. I_k^W is always radical but I_k need not be.

We will make use of Springer's theory of regular elements [16]. Given an element g of W and a complex number ζ let us write $V(g, \zeta)$ for the ζ -eigenspace of $V \xrightarrow{g} V$. The element g is called regular when it has an eigenspace $V(g, \zeta)$ containing a regular vector, and an integer k is a regular number when it is the order of a regular element, or equivalently if some $V(g, \zeta)$ contains a regular vector with ζ being a primitive k th root of unity. The following is proposition 3.2 of [16].

Lemma 1. *Let ζ be a primitive k th root of unity. Then*

$$\bigcup_g V(g, \zeta) = Z(I_k).$$

Here $Z(I_k)$ denotes the zero set of I_k , or equivalently $\bigcap_{k \nmid d_i} \{f_i = 0\}$ where f_1, \dots, f_n are basic invariants of degree d_1, \dots, d_n respectively. As Bessis observes in [2], it follows that k is regular if and only if Δ is not in I_k^W .

The other fact about regular vectors we need is the following lemma, which generalises the fact that A carries the regular representation. In fact, it provides a family of left W -module isomorphisms between A and CW parametrised by V^{reg} .

Lemma 2. *If M is any representation of W and v is a regular vector then composing with the evaluation $H \hookrightarrow S \xrightarrow{\text{ev}_v} \mathbb{C}$ at v results in an isomorphism $\text{Hom}_W(M, H) \xrightarrow{\sim} M^*$ of vector spaces.*

This is lemma 2.6 from [12]. Note that this is not surprising when $M = V$, because the quotient $V \rightarrow V//W$ is a local isomorphism at a regular vector, and the lemma simply provides an isomorphism between the two cotangent spaces.

3. THE DUALITY GROUP PAIRING

For any representation M of W there is a pairing which lands in S^W :

$$\begin{aligned} & \text{Hom}_W(M, H) \otimes \text{Hom}_W(M^*, H) \\ & \quad \downarrow \\ & \text{Hom}_W(M, S) \otimes \text{Hom}_W(M^*, S) \\ & \quad \downarrow \\ & \text{Hom}_W(M \otimes M^*, S) \\ & \quad \downarrow \\ & \text{Hom}_W(\mathbb{C}, S) = S^W \end{aligned}$$

the second arrow using multiplication in S and the third coming from the W -invariant diagonal $\mathbb{C} \rightarrow M \otimes M^*$. Note that all of these maps preserve the gradings. From S^W we can evaluate at a vector v to land in \mathbb{C} . The following diagram then commutes

$$\begin{array}{ccc} \text{Hom}_W(M, H) \otimes \text{Hom}_W(M^*, H) & \longrightarrow & S^W \\ \downarrow & & \downarrow \text{ev}_v \\ M^* \otimes M & \longrightarrow & \mathbb{C} \end{array}$$

where the leftmost map comes from applying lemma 2 to both M and M^* . If v is regular lemma 2 says this is an isomorphism. In this case going around the lower-left clearly results in a non-degenerate pairing, so we obtain:

Proposition 1. *When v is a regular vector the pairing*

$$\mathrm{Hom}_W(M, H) \otimes \mathrm{Hom}_W(M^*, H) \rightarrow S^W \xrightarrow{ev_v} \mathbb{C}$$

is perfect.

Of course this depends only on the image of v in $V//W$. Next we think about how to preserve some of the grading information so that we can relate the degrees and codegrees of M .

Lemma 2 can be restated algebraically as saying that if k is a regular number we can find a regular vector v for which evaluation at v factors through the quotient:

$$S^W \longrightarrow S^W/I_k \xrightarrow{ev_v} \mathbb{C}.$$

If one considers S^W and \mathbb{C} to be graded by $\mathbb{Z}/k\mathbb{Z}$, with \mathbb{C} in degree 0, then another way of stating this is that evaluation $S^W \rightarrow \mathbb{C}$ at v preserves this grading. Combining these remarks with proposition 1 gives the following:

Proposition 2. *Suppose k is a regular number for W . For any representation M of W there is a permutation π of $\{1, \dots, \dim M\}$ such that for each i*

$$d_i^M = d_{\pi i}^{M^*} \quad \text{modulo } k.$$

3.1. This is well-known, at least for $M = V$, and can be found for instance in [10, Theorem 11.58]. The converse is true: if such a permutation exists for $M = V$ then k is a regular number. Even stronger, it is proven in [9] that k is regular as long as it divides exactly as many degrees as codegrees.

Now we turn to the case of duality groups. This is the best situation for controlling the grading. After pairing $\mathrm{Hom}_W(V, H) \otimes \mathrm{Hom}_W(V^*, H) \rightarrow S^W$ we can canonically project onto a highest degree basic invariant

$$S^W = \mathrm{Sym}_{\mathbb{C}} \mathrm{Hom}_W(V, H) \twoheadrightarrow \mathrm{Hom}_W(V, H)_d.$$

Since A and H are canonically isomorphic, we switch freely between them below (for aesthetic reasons only).

Theorem 1. *When W is a duality group $\mathrm{Hom}_W(V, A)_d$ is one dimensional and the pairing*

$$\mathrm{Hom}_W(V, A) \otimes \mathrm{Hom}_W(V^*, A) \rightarrow \mathrm{Hom}_W(V, A)_d$$

is perfect.

Since $\mathrm{Hom}_W(V, A)_d$ is concentrated in degree d this pairing is a concrete, representation-theoretic witness to the duality $d_i + d_i^* = d$.

Proof. Firstly, $\mathrm{Hom}_W(V, H)_d$ is one dimensional because $\mathrm{Hom}_W(V^*, H)_0$ clearly is. This implies that $d \nmid d_i$ precisely for $i \neq n$. In other words, the zero-set $Z(I_d)$ of lemma 1 is $\{f_1 = \dots = f_{n-1} = 0\}$. The regularity criterion mentioned in paragraph 3.1 implies that the highest degree is always regular in a duality group. So we may invoke lemma 1 to find a regular vector v in $Z(I_d)$.

The point will be that the triangle below almost commutes, enough to make the pairing along the top perfect.

$$\begin{array}{ccc} & & \text{Hom}_W(V, H)_d \\ & \nearrow & \downarrow \wr \\ \text{Hom}_W(V, H) \otimes \text{Hom}_W(V^*, H) & \longrightarrow S^W & \downarrow \text{ev}_v \\ & \searrow & \mathbb{C} \end{array}$$

The vertical map is any choice of isomorphism, that is, a choice of highest degree basic invariant f_n (there is a canonical one, with $f_n(v) = 1$).

The lower pairing is perfect by proposition 1. Suppose $\phi \in \text{Hom}_W(V, H)$ and $\psi \in \text{Hom}_W(V^*, H)$ pair nontrivially in this way. Say $\phi \otimes \psi$ goes to $f \in S^W$ and $f(v) \neq 0$. Since $f_1(v) = \dots = f_{n-1}(v) = 0$ this f involves a monomial f_n^k . For degrees reasons, using the fact that $d_i + d_j^* < d_i + d_i^* + d_j + d_j^* = 2d$ for any i, j , we must have $k = 1$. In other words f projects nontrivially into $\text{Hom}_W(V, H)_d$, which is enough to establish the theorem. \square

In the proof we only really used that $d_1^* < d$, as we can use the result of [9] mentioned in 3.1 to get regularity of d from this. So W is a duality group if $d_1^* < d$. This inequality is clear for real reflection groups since $d_1^* = d - 2$. One can also establish directly that d is a regular number in the real case using the theory of Coxeter elements.

Once we have one regular vector in $Z(I_d)$, any nonzero vector in here will be regular. The pairing defined in terms of ev_v depends only on the orbit of v , and choosing an orbit corresponds to choosing a generator of $\text{Hom}_W(V, H)_d$. Put geometrically, $\{f_1 = \dots = f_{n-1} = 0\}$ is the preimage in V of the f_n -axis in $V//W$; by regularity it consists of $|G|/d$ lines through the origin. The coordinate ring of this set is S/I_d (the reducedness of S/I_d is equivalent to the regularity of d , since $\text{spec}(S/I_d)$ always has degree $|G|/d$ in V).

4. MATRIX FACTORISATIONS

This section contains an extremely brief introduction to the theory of matrix factorisations. A readable account of the basics of this theory can be found in [5, Section 3].

Let R be a commutative ring and f be an element of R . A matrix factorisation of f consists of a pair of projective R -modules F_0 and F_1 with a pair of R -linear maps $\phi : F_0 \rightarrow F_1$ and $\psi : F_1 \rightarrow F_0$ for which the compositions $\psi\phi$ and $\phi\psi$ are both multiplication by f . When R is graded and f is homogeneous we assume further that ϕ and ψ are homogeneous of degree zero and $|f|$ respectively.

Matrix factorisations were introduced by Eisenbud [6] to investigate the stable behaviour of modules over a hypersurface singularity. It is in this case, when R is regular, that these objects are most useful: then the homotopy category of matrix factorisations carries important geometric meaning. However, we will construct matrix factorisations over the singular ring S/I_d (over a general complete intersection, matrix factorisations control the stable behaviour of modules of complexity one).

Crucially, when f is a non-zero-divisor the 2-periodic sequence

$$\dots \rightarrow F_0 \otimes_R (R/f) \xrightarrow{\phi \otimes 1} F_1 \otimes_R (R/f) \xrightarrow{\psi \otimes 1} F_0 \otimes_R (R/f) \rightarrow \dots$$

obtained by reducing modulo f is an exact sequence of free R/f -modules. If f is homogeneous of some degree the sequence is only quasi-periodic: shifting the complex by 2 results in a grading shift by $|f|$. Let it

be emphasised that matrix factorisations do not simply provide a criterion for exactness, they are extremely important objects in their own right.

4.1. Assume again that f is a non-zero-divisor. Let F_0 and F_1 be free R -modules of equal finite rank, and assume that $\phi : F_0 \rightarrow F_1$ and $\psi : F_1 \rightarrow F_0$ are linear maps such that $\psi\phi = f \cdot 1_{F_1}$. As Eisenbud notes in [6], it is then automatically true that $\phi\psi = f \cdot 1_{F_0}$, so this data determines a matrix factorisation.

5. RESOLVING $\text{Der}_{\mathbb{C}}(A, A)$ AND $\Omega_{A/\mathbb{C}}^1$

The results of this section (especially in the Coxeter case) were the motivation for the construction of the duality group pairing.

Associated to the algebra map $S^W \rightarrow S$ there is a well-known presentation for the relative Kähler differentials $\Omega_{S^W/\mathbb{C}}^1 \otimes_{S^W} S \rightarrow \Omega_{S/\mathbb{C}}^1 \rightarrow \Omega_{S/S^W}^1 \rightarrow 0$, and in our situation $\Omega_{S^W/\mathbb{C}}^1 \otimes_{S^W} S$ is a free module by the Chevalley-Shephard-Todd Theorem. So, this descends to a free presentation

$$\Omega_{S^W/\mathbb{C}}^1 \otimes_{S^W} A \rightarrow \Omega_{S/\mathbb{C}}^1 \otimes_S A \rightarrow \Omega_{A/\mathbb{C}}^1 \rightarrow 0.$$

How can this be continued to a free resolution?

Let us phrase things dually in terms of derivations. The inclusion $S^W \rightarrow S$ gives rise to a Jacobian map $J : \text{Der}_{\mathbb{C}}(S, S) \rightarrow \text{Der}_{\mathbb{C}}(S^W, S)$. The pairing just defined allows us to construct a degree d map in the other direction, as follows. First there is a natural map given by multiplication

$$K' : \text{Hom}_W(V^*, H) \otimes S \cong (V \otimes H)^W \otimes S \hookrightarrow V \otimes (H \otimes S) \rightarrow V \otimes S.$$

And by theorem 1 we have natural isomorphisms

$$\text{Der}_{\mathbb{C}}(S^W, S) \cong \text{Hom}_W(V, H)^* \otimes S \cong \text{Hom}_W(V^*, H) \otimes S(-d).$$

We will call the composition $K : \text{Der}_{\mathbb{C}}(S^W, S) \cong \text{Hom}_W(V^*, H) \otimes S(-d) \xrightarrow{K'(-d)} \text{Der}_{\mathbb{C}}(S, S)(-d)$.

Note that all this is W -equivariant. This means, written as a matrix, the entries of JK are invariant. In particular $\det(JK)$ is an invariant polynomial which is divisible by $\det(J)$, and hence has to be divisible by Δ . Since it has the same degree as Δ , they coincide up to a scalar. Moreover, it follows that $\det(K)$ is the reduced equation for the critical locus (with K' the same argument applies to any reflection group, and this is the case $M = V$ of Gutkin's theorem [7]).

Essentially, J and K form the matrix factorisation we are after. However, differentiating introduces constants which makes this not quite true. To fix this we introduce the reduced Jacobian $\overline{J} : \text{Der}_{\mathbb{C}}(S, S) \rightarrow \text{Der}_{\mathbb{C}}(S^W, S)$ which, considered as a map $V \otimes S \rightarrow \text{Hom}_W(V, H)^* \otimes S$, is given by $\overline{J}(v \otimes f) = \sum \sigma^* \otimes \sigma(v)f$, where the sum is over a basis $\{\sigma\}$ of $\text{Hom}_W(V, H)^*$.

Let us explain why, modulo the ideal I_d , \overline{J} agrees with J up to post-composition with an invertible matrix of scalars. More precisely, if $\sigma \in \text{Hom}_W(V, H)$ and $\tilde{\sigma} = \sum u^* \sigma(u)$ is the corresponding polynomial in S^W , then we will show that $\frac{\partial \tilde{\sigma}}{\partial v} = |\sigma| \sigma(v)$ modulo I_d , so this matrix of scalars is multiplication by degree on $\text{Hom}_W(V, H)^*$. For obvious degree reasons, it suffices to do this modulo the full Hilbert ideal I . Since $S = H \oplus I$ as W representations, we can write $\frac{\partial \tilde{\sigma}}{\partial v} = h_{\sigma}(v) + i_{\sigma}(v)$ for equivariant maps $h_{\sigma} : V \rightarrow H$ and $i_{\sigma} : V \rightarrow I$. Since $\sum v^* \frac{\partial \tilde{\sigma}}{\partial v} = |\sigma| \tilde{\sigma}$ the map $k_{\sigma} = |\sigma| h_{\sigma} - \sigma : V \rightarrow H$ satisfies $\sum v^* k_{\sigma}(v) = |\sigma| \sum v^* i_{\sigma}(v)$, but this is a decomposable invariant polynomial (by applying the Reynolds operator to the right-hand-side),

and the Chevalley-Sheppard-Todd theorem says the map $\text{Hom}_W(V, H) \rightarrow S^W$ is injective and has only indecomposables in its image, hence $k_\sigma = 0$.

In particular, $\overline{J} \otimes_S A$ has the same kernel as $J \otimes_S A$, and any matrix factorisation involving $\overline{J} \otimes_S S/I_d$ is isomorphic to one involving $J \otimes_S S/I_d$.

Theorem 2. *Let W be a duality group equipped with a choice of highest degree basic invariant f_n . The maps \overline{J} and K defined above*

$$\text{Der}_{\mathbb{C}}(S, S) \xrightleftharpoons{\quad} \text{Der}_{\mathbb{C}}(S^W, S)$$

result in a matrix factorisation of f_n after reducing modulo f_1, \dots, f_{n-1} . Hence applying $- \otimes_S A$ we obtain a 2-periodic minimal free resolution of $\text{Der}_{\mathbb{C}}(A, A)$. Dualizing, we also obtain a 2-periodic minimal free resolution of $\Omega_{A/\mathbb{C}}^1$.

Proof. A straight-forward calculation verifies that $\overline{J}K = f_n \cdot 1_{\text{Der}_{\mathbb{C}}(S^W, S/I_d)}$, it goes as follows. Fix an element ∂ of $\text{Hom}_W(V, H)^*$, considered as a derivation in $\text{Der}_{\mathbb{C}}(S^W, S)$. The corresponding element ϕ of $\text{Hom}_W(V^*, H)$ by definition satisfies $\sum_v \phi(v^*)\sigma(v) = \partial(\sigma)f_n$ after reducing to S/I_d , where the sum is taken over some basis $\{v\}$ of V , and σ is any element of $\text{Hom}_W(V, H)$. The map defined above simply takes ϕ to $K(\partial) = \sum v \otimes \phi(v^*)$ now considered as an element of $V \otimes S \cong \text{Der}_{\mathbb{C}}(S, S)$. The reduced Jacobian then takes $\sum v \otimes \phi(v^*)$ to

$$\overline{J}K(\partial) = \sum_{\sigma} \sum_v \sigma^* \otimes \sigma(v)\phi(v^*) = \sum_{\sigma} \sum_v \sigma^* \otimes \partial(\sigma)f_n$$

for some basis $\{\sigma\}$ of $\text{Hom}_W(V, H)$. From here we see $\overline{J}K(\partial)(\sigma) = \partial(\sigma)f_n$ holds in S/I_d . After this, the observation of paragraph 4.1 completes the proof that \overline{J} and K determine a matrix factorisation of f_n .

The remaining assertions follow from the presentation of $\text{Der}_{\mathbb{C}}(A, A)$ as the kernel of $J : \text{Der}_{\mathbb{C}}(S, A) \rightarrow \text{Der}_{\mathbb{C}}(S^W, A)$ and of $\Omega_{A/\mathbb{C}}^1$ as the cokernel of $J^* : \Omega_{S^W/\mathbb{C}}^1 \otimes_{S^W} A \rightarrow \Omega_{S/\mathbb{C}}^1 \otimes_S A$. \square

The appearance of these matrix factorisations is quite surprising. There is no obvious reason that the A -module $\Omega_{A/\mathbb{C}}^1$ should have complexity one, this is far from the generic situation.

As remarked above, up to a scalar $\det(JK)$ is Δ , so we recover the fact that when W is a duality group, Δ is monic in f_n of degree n .

Example. Already interesting is the case of the symmetric group \mathfrak{S}_{n+1} acting on $V = \{(\alpha_0, \dots, \alpha_n) : \sum \alpha_i = 0\}$. Then $S = \mathbb{C}[x_0, \dots, x_n]/(\sum x_i = 0)$ where x_i is dual to $(0, \dots, 1, \dots, 0)$. It is classical that $S^{\mathfrak{S}_{n+1}}$ is the polynomial ring $\mathbb{C}[\sigma_2, \dots, \sigma_{n+1}]$ on the elementary symmetric polynomials $\sigma_k = (-1)^{k-1} \sum_{i_1 < \dots < i_k} x_{i_1} \cdots x_{i_k}$. So the algebra of coinvariants is $A = S/(\sigma_2, \dots, \sigma_{n+1})$, and the highest degree is $d = n + 1$. The duality pairing will naturally match σ_{i+2} with σ_{n+1-i} . We need to choose bases to get matrices. If $v = \frac{1}{n+1}(1, \dots, 1)$ and $v_i = (0, \dots, 1, \dots, 0) - v$ then when we differentiate we get $\frac{\partial \sigma_k}{\partial v_i} = x_i^{k-1}$ modulo $\sigma_2, \dots, \sigma_n$. So we take as bases v_1, \dots, v_n for V and $\sigma_2, \dots, \sigma_{n+1}$ for $\text{Hom}_{\mathfrak{S}_{n+1}}(V, H)$, and we get

$$J = \begin{pmatrix} x_1 & \cdots & x_n \\ \vdots & & \vdots \\ x_1^n & \cdots & x_n^n \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} x_1^n - x_0^n & \cdots & x_1 - x_0 \\ \vdots & & \vdots \\ x_n^n - x_0^n & \cdots & x_n - x_0 \end{pmatrix}$$

as matrices with entries in S/I_d (calculating K is made easier by the observation below). Even knowing these matrices, it is combinatorially quite involved to check directly that they form a matrix factorisation of σ_{n+1} .

Going back to an arbitrary duality group, we can read from the matrix factorisation of theorem 2 the precise dimensions of all the graded components of $\Omega_{A/\mathbb{C}}^1$ and $\text{Der}_{\mathbb{C}}(A, A)$. As well as being the degree of the socle of A , recall that $N = (d_1 - 1) + \cdots + (d_n - 1)$ is the number of nontrivial reflections in W .

Theorem 3. *The Hilbert series of the graded modules $\Omega_{A/\mathbb{C}}^1$ and $\text{Der}_{\mathbb{C}}(A, A)$ are given by*

$$H_{\Omega_{A/\mathbb{C}}^1}(t) = \left(\sum_{i \leq n} \frac{t - t^{d_i}}{1 - t} \right) \cdot \left(\prod_{i < n} \frac{1 - t^{d_i}}{1 - t} \right),$$

$$H_{\text{Der}_{\mathbb{C}}(A, A)}(t) = \left(\sum_{i \leq n} \frac{t^{d_i^*} - t^{d-1}}{1 - t} \right) \cdot \left(\prod_{i < n} \frac{1 - t^{d_i}}{1 - t} \right).$$

In particular the total dimension is

$$\dim_{\mathbb{C}} \Omega_{A/\mathbb{C}}^1 = \frac{N|W|}{d} = \dim_{\mathbb{C}} \text{Der}_{\mathbb{C}}(A, A).$$

Proof. We'll first do this for the Kähler differentials. From the exact sequence

$$0 \rightarrow \ker(J^*) \rightarrow \Omega_{S^W/\mathbb{C}}^1 \otimes_S A \xrightarrow{J^*} \Omega_{S/\mathbb{C}}^1 \otimes_S A \rightarrow \text{coker}(J^*) \rightarrow 0,$$

there is an equality of Hilbert series

$$H_{\Omega_{S/\mathbb{C}}^1 \otimes_S A} - H_{\Omega_{S^W/\mathbb{C}}^1 \otimes_S A} = H_{\text{coker}(J^*)} - H_{\ker(J^*)}.$$

But $\text{coker}(J^*) = \Omega_{A/\mathbb{C}}^1$ and the matrix factorisation gives us an isomorphism $\ker(J^*) \cong \text{coker}(J^*)(d)$. Also note that $H_{\Omega_{S/\mathbb{C}}^1 \otimes_S A}(t) = ntH_A(t)$ and $H_{\Omega_{S^W/\mathbb{C}}^1 \otimes_S A}(t) = (\sum_i t^{d_i})H_A(t)$. From this we deduce

$$ntH_A(t) - (\sum_i t^{d_i})H_A(t) = H_{\Omega_{A/\mathbb{C}}^1}(t) - t^d H_{\Omega_{A/\mathbb{C}}^1}(t).$$

Keeping in mind that $H_A(t) = \prod_i \frac{1-t^{d_i}}{1-t}$, this can be rearranged to give the expression of the theorem.

We can deduce the expression for $H_{\text{Der}_{\mathbb{C}}(A, A)}$ using the equation $H_{\text{Hom}_A(M, A)}(t) = t^{(d_1-1)+\cdots+(d_n-1)} \cdot H_M(t^{-1})$, which holds for any graded module since A is a Frobenius algebra with socle in degree $(d_1-1)+\cdots+(d_n-1)$.

Finally, setting $t = 1$ in either polynomial gives $[(d_1-1)+\cdots+(d_n-1)] \cdot d_1 \cdots d_{n-1} = \frac{N|W|}{d}$. \square

When W is a real reflection group the matrix factorisation takes a particularly striking form. In this case we may equip V with a perfect W -invariant bilinear (as opposed to linear-antilinear) form $V \otimes V \rightarrow \mathbb{C}(-2)$. There is canonical one when W is the Weyl group of a flag manifold, for instance. We obtain a $\mathbb{C}W$ -isomorphism $V(2) \cong V^*$, and hence an isomorphism

$$\Omega_{S/\mathbb{C}}^1 \cong V^* \otimes S \cong V \otimes S(2) \cong \text{Der}_{\mathbb{C}}(S, S)(2).$$

More interestingly, choosing a highest degree basic invariant f_n gives us an isomorphism

$$\begin{aligned} \Omega_{S^W/\mathbb{C}}^1 &\cong \text{Hom}_W(V, H) \otimes S^W \cong \text{Hom}_W(V^*, H)^* \otimes S^W(d) \cong \\ &\text{Hom}_W(V, H)^* \otimes S^W(d+2) \cong \text{Der}_{\mathbb{C}}(S^W, S^W)(d+2). \end{aligned}$$

Dual to the reduced Jacobian is the canonical map $\overline{J}^* : \Omega_{S^W/\mathbb{C}}^1 \otimes_{S^W} S \rightarrow \Omega_{S/\mathbb{C}}^1$, which up to an invertible matrix agrees with the usual J^* . Under the identifications just described this is actually the map $K : \text{Der}_{\mathbb{C}}(S^W, S) \rightarrow \text{Der}_{\mathbb{C}}(S, S)$ of proposition 2. Hence

Corollary 1. *Let W be a real reflection group equipped with a perfect W -invariant bilinear form and a choice of highest degree basic invariant f_n . Making the above identifications, the reduced Jacobian and its dual:*

$$\begin{array}{ccc} \Omega_{S^W/\mathbb{C}}^1 \otimes_{S^W} S & \longrightarrow & \Omega_{S/\mathbb{C}}^1 \\ \parallel \wr & & \parallel \wr \\ \text{Der}_{\mathbb{C}}(S^W, S) & \longleftarrow & \text{Der}_{\mathbb{C}}(S, S) \end{array}$$

determine a matrix factorisation of f_n when reduced to S/I_d . In particular, the 2-periodic resolution of $\Omega_{A/\mathbb{C}}^1$ and $\text{Der}_{\mathbb{C}}(A, A)$ thus obtained is isomorphic to its own dual, shifted by one.

We obtain the following curious fact. Recall that the first syzygy $\text{Syz}_A^1 M$ of a graded A module M is by definition the kernel of a surjection $A^k \rightarrow M$ from a free module with the minimal possible number of generators.

Corollary 2. *If A is the algebra of coinvariants associated to a Coxeter group then*

$$\text{Syz}_A^1 \Omega_{A/\mathbb{C}}^1 = \text{Der}_{\mathbb{C}}(A, A)(d+2) \quad \text{and} \quad \text{Syz}_A^1 \text{Der}_{\mathbb{C}}(A, A) = \Omega_{A/\mathbb{C}}^1(-2).$$

Moreover we have

$$\dim_{\mathbb{C}} \Omega_{A/\mathbb{C}}^1 = \frac{n|W|}{2} = \dim_{\mathbb{C}} \text{Der}_{\mathbb{C}}(A, A).$$

Finally, let us point out that, being a complete intersection, the André-Quillen cohomology of A can be computed as the cohomology of the complex $\text{Der}_{\mathbb{C}}(S, A) \xrightarrow{J} \text{Der}_{\mathbb{C}}(S^W, A)$, which lives in cohomological degree 0 and 1. From the matrix factorisation we get a canonical isomorphism $H_{AQ}^0(A, A) \cong H_{AQ}^1(A, A)(d)$, so from theorem 3 we can read off the entire bigraded Hilbert series of the André-Quillen cohomology.

In this context, the first André-Quillen cohomology group is also known as the Tjurina module $H_{AQ}^1(A, A) = T_A^1 = \text{coker}(J)$. The dimension $\dim_{\mathbb{C}} T_A^1$ is the Tjurina number of the singularity defined by A . This is the number of parameters in the base of the miniversal deformation of A (or, of the singularity it defines). This is explained in [11, chapter 6]. Theorem 2 produces a canonical isomorphism $\text{Der}_{\mathbb{C}}(A, A) \cong T_A^1(d)$, hence the Tjurina number of A is $\frac{N|W|}{d}$ (which is $\frac{n|W|}{2}$ in the Coxeter case). Moreover, this allows one to produce a basis for T_A^1 (this calculation is entirely feasible in the case of the symmetric group, for instance), and from this data one can explicitly construct the miniversal deformation of A , see loc. cit.. It would be interesting to try to identify certain natural deformations of A in this space.

6. FLAG MANIFOLDS

A number of authors have investigated the rational homotopy types of flag manifolds, especially what can be said about classifying their self-maps, e.g [15, 14]. The purpose of this section is to briefly point out that the calculations above can be interpreted in these terms.

Thus, let $X = G/B$ be a complete flag manifold for some semi-simple complex algebraic group G and Borel subgroup B . Let T be a maximal torus in B of rank n and set $V = \text{Lie}(T)$. In this situation we should place V in cohomological degree -2 , doubling the gradings from the previous sections. The associated Weyl

group W acts on V as a duality group, so the above results apply. According to the famous Borel picture there is a canonical isomorphism

$$A = \mathbb{C}[V]/\mathbb{C}[V]_+^W \xrightarrow{\cong} H^*(X; \mathbb{C}).$$

Because X is a formal space, the graded algebra A completely determines the rational homotopy type of X , and this has been well exploited in the literature. For instance, there is an isomorphism of (shifted) Lie algebras⁴

$$\pi_*(X)_{\mathbb{C}} \cong H_{\text{AQ}}^*(A, \mathbb{C}) \cong V \oplus \text{Hom}_W(V, A)(1).$$

Where $\pi_*(X)_{\mathbb{C}} = \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{C}$ is the shifted (and complexified) homotopy Lie algebra. The first isomorphism can be found in for example in [4], the second isomorphism is a standard computation of the André-Quillen cohomology of a complete intersection. This calculation of the rational homotopy groups of a flag manifold is presumably well-known.

An easy consequence of the previous section is that the Halperin conjecture holds for flag manifolds: the Serre spectral sequence associated to any fibration with fibre X must collapse at the second page. This is known to be equivalent to the fact that $\text{Der}_{\mathbb{C}}(A, A)_{<0} = 0$, which can be read from theorem 3. This was proven by Shiga and Tezuka in [14] (but actually has been known for any simply connected Kähler manifold since [3]).

Denote by $\text{aut}(X, 1)$ the component of the identity in the mapping space $\text{map}(X, X)$. There is also an isomorphism

$$\pi_{>0}(\text{aut}(X, 1))_{\mathbb{C}} \cong H_{\text{AQ}}^{<0}(A, A),$$

which can be found in [13], or in this context this context see [8, Theorem B]. In this way Meier describes the dimensions of $\pi_i(\text{aut}(X, 1))_{\mathbb{C}}$ in terms of derivations on A . We can be more explicit:

Theorem 4. *For $i \geq 1$, the dimension of $\pi_i(\text{aut}(X, 1))_{\mathbb{C}}$ is the coefficient of t^{-i} in the Laurent polynomial*

$$\left(\sum_{i \leq n} \frac{t^{1-2d_i} - t^{-1}}{1 - t^2} \right) \cdot \left(\prod_{i < n} \frac{1 - t^{2d_i}}{1 - t^2} \right).$$

Any topological monoid is rationally homotopy equivalent to a product of Eilenberg-MacLane spaces, hence the above theorem completely determines the rational homotopy type of $\text{aut}(X, 1)$; it is a product of odd dimensional (rational) spheres, the number in each dimension being given by the coefficients of above Laurent polynomial. In [15] Smith also calculates the rational homotopy type of $\text{aut}(X, 1)$ (at least for $GL(n)$ and $Sp(n)$). His expression for the number of spheres looks quite different.

It may be of interest to topologists to have a full description of $\text{Der}_{\mathbb{C}}(H^*(G/B; \mathbb{C}), H^*(G/B; \mathbb{C}))$.

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⁴The homological grading on the left is the negative of the cohomological grading on the right. We have also totalised the internal and cohomological gradings on the right by adding them together.

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